Discrete torsion and twisted orbifold cohomology

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1 Introduction

One of the remarkable insights of orbifold string theory is an indication of the existence of a new cohomology theory of orbifolds containing so-called twisted sectors as the contribution of singularities. Mathematically, such an orbifold cohomology theory has been constructed by Chen-Ruan [CR]. Author believes that there is a "stringy" geometry and topology of orbifolds of which orbifold cohomology is its core. One aspect of this new geometry and topology is the twisted orbifold cohomology and its relation to discrete torison. Let me first explain their physical origin. Physicists usually work over a global quotient $X = Y/G$ only, where G is a finite group acting smoothly on Y. A discrete torsion is a cohomology class $\alpha \in H^2(G, U(1))$. Physically, a discrete torsion counts the freedom to choose a phase factor to weight path integral over each twisted sector without destroying the consistency of string theory. For each α, Vafa-Witten [VW] constructed the twisted orbifold cohomology group $H^*_{orb,\alpha}(X/G, \mathbf{C})$.

Mathematically, Vafa-Witten suggested that discrete torsion and twisted orbifold cohomology is connected to the problem of desingularizations. Recall that there are two methods to remove singularities, resolution or deformation. Both play important roles in the theory of Calabi-Yau 3-folds. One can obtain a smooth manifold Y from an orbifold X by using a combination of resolution and deformation. We call Y a desingularization of X. In string theory, we also require the resolution to be a crepant resolution. It is known that a desingualization may not exist in dimension higher than three. In this case, we allow our desingularization to be an orbifold. As we mentioned in [CR], physicists predicted that ordinary orbifold cohomology group is the same as ordinary cohomology group of its crepant resolution. Vafa-Witten suggested that discrete torsion is a parameter for deformation. Furthermore, the cohomology of the desingularization is the twisted orbifold cohomology of discrete torsion plus possible contributions of exceptional loci of small resolution. A small resolution is a special kind of resolution whose exceptional loci is of codimension 2 or more. However, this proposal immediately ran into trouble because there are many more desingularizations than the number of discrete torsions. For example, D. Joyce [JO] constructed five different desingularizations of $T^6/\mathbf{Z}_4$ while $H^2(\mathbf{Z}_4, U(1)) = 0$. To count these "missing" desingularizations seems to be a serious problem. On the another hand, it is well-known that most orbifolds (even Calabi-Yau orbifolds) are not global quotients. Therefore, it is also necessary to develop the theory over general orbifolds.

We will address both problems in this paper. First, we introduce the notion of inner local system $\mathcal{L}$ for arbitrary orbifold. A local system is defined as an assignment of a flat (orbifold)-line bundle $L_{(g)}$ to each sector $X_{(g)}$ satisfying certain compatibility condition (See definition 2.1). Such a compatibility condition is designed in such a way that Poincare duality and cup product in ordinary orbifold cohomology survive the process of twisting. Then, twisted orbifold cohomology $H^*_{orb}(X, \mathcal{L})$ is defined as orbifold cohomology with value in the inner local system (See Definition 2.2). We will demonstrate that our inner local systems count all the examples constructed by D. Joyce. The author believes that the inner local system is a more fundamental notion than

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the discrete torsion. Then, we can formulate following mathematical conjecture: *Suppose that X is a Calabi-Yau Gorenstein orbifold. For every desingularization, we can associate a inner local systems such that as additive groups the ordinary orbifold cohomology of desigularization is the sum of twisted orbifold cohomology and contributions from exceptional loci of small resolution.*

Our next goal is to determine appropriate notion of discrete torsion for general orbifold. Let X be an arbitrary almost complex orbifold. The author's key observation is that we should use the orbifold fundamental group $\pi_1^{orb}(X)$ (See definition 2.1) to replace G. Then a discrete torsion of X is defined as a cohomology class $\alpha \in H^2(\pi_1^{orb}(X), U(1))$. Note that if $X = Y/G$ is a global quotient, there is a short exact sequence

$$1 \to \pi_1(Y) \to \pi_1^{orb}(X) \to G \to 1. \tag{1.1}$$

It induces a homomorphism $H^2(G, U(1)) \to H^2(\pi_1^{orb}(X), U(1))$. Hence a discrete torsion in the sense of Vafa-Witten induces the discrete torsion in this paper. In fact, we can do better, we can define a local discrete torsion for each connected component of singular loci. A global discrete torsion is defined as an assignment of a local discrete torsion to a connected component of singular loci. Then, the link between discrete torsion and twisted orbifold cohomology is the theorem that a global discrete torsion induces an inner local system and hence define a twisted orbifold cohomology. However, we want to emphasis that not every inner local system comes from discrete torsion (See example 5.3).

We will introduce inner local system and twisted orbifold cohomology ring in section 2. The section 3 is devoted to discrete torsion. The relation between discrete torsion and local system is discussed in section 4. Finally, some examples are computed in section 5. This paper can be viewed as a sequel to [CR]. Since many constructions are similar, we will follow the notations of [CR] and be sketchy in the details. The author strongly encourages readers to read [CR] first before reading this paper.

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2 Local system and twisted orbifold cohomology

2.1 Review of ordinary orbifold cohomology

Suppose that X is an orbifold. By the definition, X is a topological space with a system of orbifold charts (uniformizing system). Namely, every point $p \in X$ has a system of orbifold chart of the form U_p/G_p where U_p is a smooth manifold and G_p is a finite group acting on U_p fixing p. G_p is called a local group. Note that the action of G_p does not have to be effective. If it does, we call it a reduced orbifold. We use (U_p, G_p) to denote the chart. The patching condition is follows: if $q \in U_p/G_p \cap U_r/G_r$, there is an orbifold chart $U_q/G_q \subset U_p/G_p \cap U_r/G_r$. Moreover, the inclusion map $i : U_q/G_q \subset U_p/G_p$ can be lifted to a smooth map

$$\tilde{i}_{pq} : U_q \to U_p \tag{2.1}$$

and an injective homomorphism

$$i_{\#,pq} : G_q \to G_p \tag{2.2}$$

such that $\tilde{i}$ is $i_{\#}$-equivariant.

$$i_{pq} = (\tilde{i}, i_{\#}) : (U_q, G_q) \to (U_p, G_p)$$

is called an injection. A different lifting differs from $\tilde{i}$ by the action of an element of G_p. Moreover, $i_{\#}$ differs by the conjugation of the same element. We say that the corresponding injections are equivalent.

Therefore, for any $g \in G_q$, the conjugacy class $(i_{\#}(g))_{G_p}$ is well-defined. We define an equivalence relation $(g)_{G_q} \cong (i_{\#}(g))_{G_p}$. Let T_1 be the set of equivalence classes. By abusing the notation, we use (g) to denote the equivalence class to which $(g)_{G_q}$ belongs to. For each $(g) \in T_1$, we can define a sector

$$X_{(g)} = \{(x, (g')_{G_x}) | g' \in G_x, (g')_{G_x} \in (g)\}. \tag{2.3}$$

It was shown in [CR] that $X_{(g)}$ is an orbifold. It is the common convention that we call $X_{(g)}$ for $g \neq 1$ a twisted sector and $X_{(1)}$ a nontwisted sector. Once we define sectors, we diagonalize the action of g in $T_x X_{(g)}$ for each $x \in X_{(g)}$. Suppose that $g = diag(e^{\frac{2\pi i m_1}{m}}, \cdots, e^{\frac{2\pi i m_n}{m}})$, where m is the order of g and $0 \leq \frac{m_i}{m} < 1$. Then we define the degree shifting number $\iota_{(g)} = \sum_i \frac{m_i}{m}$. One can show that $\iota_{(g)}$ is independent of $x \in X_{(g)}$. The ordinary orbifold cohomology is defined as

$$H^*_{orb}(X, \mathbf{C}) = \oplus_{(g) \in T_1} H^{*-2\iota_{(g)}}(X_{(g)}, \mathbf{C}). \tag{2.4}$$

There is a diffeomorphism $I : X_{(g)} \to X_{(g^{-1})}$ defined by $(x, (g)) \to (x, (g^{-1}))$. Poincare paring $< >_{orb}$ of orbifold cohomology is defined as the direct sum of

$$< >^{(g)}_{orb}: H^{d-2\iota_{(g)}}(X_{(g)}, \mathbf{C}) \otimes H^{2n-d-2\iota_{(g^{-1})}}(X_{(g^{-1})}, \mathbf{C}) \to \mathbf{C} \tag{2.5}$$

defined by

$$< \alpha, \beta >^{(g)}_{orb} = \int_{X_{(g)}} \alpha \wedge I^* \beta. \tag{2.6}$$

Next, we consider cup product. We first construct a moduli space (see [CR] (section 4.1)).

$$X_3 = \{(x, (g_1, g_2, g_3)_{G_x}) | g_i \in G_x, g_1 g_2 g_3 = 1\} \tag{2.7}$$

X_3 is an orbifold. Let $\mathbf{g} = (g_1, \cdots, g_k)$ with $g_i \in G_q$. By abusing the notation, we simply say $\mathbf{g} \in G_q$. We define the equivalence relation $(\mathbf{g})_{G_q} \cong (i_{\#}(\mathbf{g}))_{G_q}$. Let T_k be the set of equivalence class and use $(\mathbf{g})$ to denote the equivalence class such that $(\mathbf{g})_{G_q} \in (\mathbf{g})$. We will use $T^o_k \subset T_k$ to denote the set of equivalence classes of $(\mathbf{g})$ such that $\mathbf{g} = (g_1, \cdots, g_k)$ with $g_1 \cdots g_k = 1$. It was proved in [CR] that

$$X_{(\mathbf{g})} = \{(x, (\mathbf{g}')_{G_x}) | \mathbf{g}' \in G_x, (\mathbf{g}') = (\mathbf{g})\}. \tag{2.8}$$

is an orbifold. One can check that

$$X_3 = \bigsqcup_{(\mathbf{g}) \in T^o_3} X_{(\mathbf{g})}, \tag{2.9}$$

Then, for each $(\mathbf{g}) \in T^o_3$, we can define evaluation maps

$$e_i : X_{(\mathbf{g})} \to X_{(g)}$$

by

$$e_i(x,(g_1,g_2,g_3)_{G_x})=(x,(g_i)_{G_x}).$$

Furthermore, there is an obstruction bundle E (see Lemma 4.2.2 [CR]). Then, we can define a three-point function

$$<\alpha,\beta,\gamma>_{orb}=\int_{X_{(\mathbf{g})}} e_1^*\alpha\wedge e_2^*\beta\wedge e_3^*\gamma\wedge e(E), \tag{2.10}$$

for any $\alpha\in H^{p-\iota(g_1)}(X_{(g_1)},\mathbf{C}),\beta\in H^{q-\iota(g_2)}(X_{(g_2)},\mathbf{C}),\gamma\in H^{2n-p-q-\iota(g_3)}(X_{(g_3)},\mathbf{C})$. Once the three point function is defined, the cup product is defined by the equation

$$<\alpha\cup_{orb}\beta,\gamma>_{orb}=<\alpha,\beta,\gamma>_{orb}. \tag{2.11}$$

for arbitrary γ.

There is a Dolbeaut version of orbifold cohomology ring (Dolbeaut orbifold cohomology ring) with identical construction. We refer reader to [CR] for detail.

Next, we consider the bundle over each sector and its pull-back. By [CR1], this is a very subtle problem and one has to be careful. We will give explicit construction in our case and refer reader to general theory in [CR1].

Now, let's examine the orbifold structure of twisted sectors more carefully. Suppose that $p\in X_{(g)}$ and an orbifold chart of $p\in X$ is U_p/G_p. By [CR1](Lemma 3.1.1), an orbifold chart of $X_{(g)}$ can be described as follows. Choose a representative of $(g)_{G_p}$, say g_p. Then, a local orbifold chart of p is $U_{g_p}/C(g_p)$, where U_{g_p} is the fixed point loci of g_p and $C(g_p)$ is the centralizer. In general, $C(g_p)$ may not acts freely on U_{g_p}. The patching map of $X_{(g)}$ is defined in the same way. More generally, $X_{(\mathbf{g})}$ also has a structure of a orbifold given by $(U_{\mathbf{g}_p},C(\mathbf{g}_p))$, where $U_{\mathbf{g}_p},C(\mathbf{g}_p)$ are the fixed point loci and centralizer of $\mathbf{g}_p$. We denote the corresponding patching map by injection $i_{\mathbf{g},pq}=(\tilde{i}_{pq,(\mathbf{g})},i_{\#,pq,(\mathbf{g})})$.

Now, an orbifold-bundle $f:E\rightarrow X_{(\mathbf{g})}$ is a continuous map between topological space such that E has a structure of orbifold as follows. Suppose that $p\in X_{(\mathbf{g})}$. E is covered by chart of the form $(U_{\mathbf{g}_p}\times\mathbf{R}^n,C(\mathbf{g}_p))$ such that the restriction of f is the projection $U_{\mathbf{g}_p}\times\mathbf{R}^n\rightarrow U_{\mathbf{g}_p}$ equivariant under $C(\mathbf{g}_p)$. For any injection $i:(U_{\mathbf{g}_q},C(\mathbf{g}_q))\rightarrow(U_{\mathbf{g}_p},C(\mathbf{g}_p))$, there is an injection between charts $(U_{\mathbf{g}_q}\times\mathbf{R}^n,G_{\mathbf{g}_q})\rightarrow(U_{\mathbf{g}_p}\times\mathbf{R}^n,G_{\mathbf{g}_p})$ given by $(\tilde{i}_{pq,(\mathbf{g})}\times g_i,i_{\#,pq,(\mathbf{g})})$, where $g_i:U_{\mathbf{g}_q}\rightarrow Aut(\mathbf{R}^n)$ satisfies the condition $g_{j\circ i}=g_j\circ g_i$. The last condition is to ensure that the equivalent injections on $X_{(\mathbf{g})}$ implies the equivalence between corresponding injections on total space of bundle.

Next, we consider evaluation map $e_i:X_{(\mathbf{g})}\rightarrow X_{(g)}$. A crucial observation is that we can choose the orbifold charts of $X_{(\mathbf{g})},X_{(g)}$ silmontaneously. Suppose that $p\in X_{(\mathbf{g})}$. We first choose $\mathbf{g}_p$. Then, it gives a natural choice for $g_{i,p}$. Similarly, an injection $i_{pq,(\mathbf{g})}$ between the charts of $X_{(\mathbf{g})}$ gives a natural choice of injection $\lambda(i_{pq,(\mathbf{g})})=i_{pq,(g_i)}$ with the property $\lambda(j\circ i)=\lambda(j)\circ\lambda(i)$. The evaluation map is interpreted as an inclusion

$$e_{i,p}:U_{\mathbf{g}_p}\rightarrow U_{g_{i,p}},e_{\#,i,p}:C(\mathbf{g}_p)\rightarrow C(g_{i,p}). \tag{2.11}$$

Following [CR1], we say that e_i is a good map. By [CR1], if $E\rightarrow X_{(g_i)}$ is a orbifold-bundle, $e^*E\rightarrow X_{(\mathbf{g})}$ is a orbifold-bundle.

Note that the direct sum and tensor product of orbifold-bundles is still a orbifold bundle. Moreover, all the differential geometric constructions such as differential form, connection and curvature work over orbifold-bundle.

2.2 Inner Local system and twisted orbifold cohomology ring

Now, we introduce the notion of inner local system for orbifold.

Definition 2.1: *Suppose that X is an orbifold (almost complex or not). An inner local system $\mathcal{L} = \{L_{(g)}\}_{g\in T_1}$ is an assignment of a flat complex orbifold line bundle over*

$$L_{(g)} \to X_{(g)}$$

to each sector $X_{(g)}$ satisfying the compatibility condition

(1) $L_{(1)} = 1$ *is trivial.*

(2) $I^* L_{(g^{-1})} = L_{(g)}^{-1}$.

(3) *Over each $X_{(\mathbf{g})}$ with $(\mathbf{g}) \in T_3^o$, $\otimes_i e_i^* L_{(g_i)} = 1$.*

If X is a complex orbifold, we assume that $L_{(g)}$ is holomorphic.

Definition 2.2: *Given an inner local system $\mathcal{L}$, we define the twisted orbifold cohomology*

$$H^*_{orb}(X, \mathcal{L}) = \oplus_{(g)} H^{*-2\iota_{(g)}}(X_{(g)}, L_{(g)}).$$

Definition 2.3: *Suppose that X is a closed complex orbifold and $\mathcal{L}$ is an inner local system. We define Dolbeault cohomology groups*

$$H^{p,q}_{orb}(X, \mathcal{L}) = \oplus_{(g)} H^{p-\iota_{(g)}, q-\iota_{(g)}}(X_{(g)}; L_{(g)}). \tag{2.12}$$

Proposition 2.4: *If X is a Kahler orbifold, we have Hodge decomposition*

$$H^k_{orb}(X, \mathcal{L}) = \oplus_{k=p+q} H^{p,q}_{orb}(X, \mathcal{L}). \tag{2.13}$$

Proof: Note that each sector $X_{(g)}$ is a Kähler orbifold. The proposition follows by applying the ordinary Hodge theorem with twisted coefficients to each sector $X_{(g)}$. $\Box$

The property (2) of Definition 2.1 can be used to show that Poincare pairing defined in (2.6) can be adopted to twisted orbifold cohomology.

Definition (Poincaré duality) 2.5: *Suppose that X is a $2n$-dimensional closed almost complex orbifold. We define a pairing*

$$< \ >_{orb,\mathcal{L}} : H^d_{orb}(X, \mathcal{L}) \otimes H^{2n-d}_{orb}(X, \mathcal{L}) \to \mathbf{C}. \tag{2.14}$$

as the direct sum of

$$< \ >^{(g)}_{orb,\mathcal{L}} : H^{d-2\iota_{(g)}}(X_{(g)}, L_{(g)}) \otimes H^{2n-d-2\iota_{(g^{-1})}}(X_{(g^{-1})}, L_{(g^{-1})}) \to \mathbf{C} \tag{2.15}$$

defined by

$$<\alpha, \beta>^{(g)}_{orb,\mathcal{L}} = \int_{X_{(g)}} \alpha \wedge I^*\beta. \tag{2.16}$$

Note that $L_{(g)}I^*L_{(g^{-1})} = 1$. Hence the integral (2.6) makes sense.

Theorem 2.6: *The pairing $< >_{orb,\mathcal{L}}$ is nondegenerate.*

Proof: The proof follows from ordinary Poincare duality on $X_{(g)}$ with twisted coefficient.

There is also a version of Poincaré duality for twisted Dolbeault cohomology. Suppose that X is a closed complex orbifold of complex dimension n. Then $X_{(g)}$ is a closed complex orbifold.

Definition 2.7: *We define a pairing*

$$< >_{orb,\mathcal{L}} : H^{p,q}_{orb}(X,\mathcal{L}) \otimes H^{n-p,n-q}_{orb}(X,\mathcal{L}) \to \mathbf{C}. \tag{2.17}$$

as the direct sum of

$$< >^{(g)}_{orb,\mathcal{L}} : H^{p-\iota_{(g)},q-\iota_{(g)}}(X_{(g)},L_{(g)}) \otimes H^{n-p-\iota_{(g^{-1})},n-q-\iota_{(g^{-1})}}(X_{(g^{-1})},L_{(g^{-1})}) \to \mathbf{C} \tag{2.18}$$

defined by

$$< \alpha, \beta >^{(g)}_{orb,\mathcal{L}} = \int_{X_{(g)}} \alpha \wedge I^*\beta. \tag{2.19}$$

Theorem 2.8: *The pairing (2.17) is nondegenerate.*

The property (3) of Definition 2.1 shows that the integral (2.10) makes sense for twisted orbifold cohomology classes. The same construction of [CR] goes through without change. We can define a twisted orbifold product $\cup_{orb,\mathcal{L}}$ in the same fashion. The same proof as in [CR] yields

Theorem 2.9: *Let X be a closed almost complex orbifold with almost complex structure J and $\dim_{\mathbf{C}} X = n$. There is a cup product $\cup_{orb,\mathcal{L}} : H^p_{orb}(X;\mathcal{L}) \times H^q_{orb}(X;\mathcal{L}) \to H^{p+q}_{orb}(X;\mathcal{L})$ for any $0 \le p,q \le 2n$ such that $p+q \le 2n$, which has the following properties:*

1. *The total twisted orbifold cohomology group $H^*_{orb}(X;\mathcal{L}) = \oplus_{0\le d\le 2n} H^d_{orb}(X;\mathcal{L})$ is a ring with unit $e^0_X \in H^0_{orb}(X;\mathcal{L})$ under $\cup_{orb,\mathcal{L}}$, where e^0_X is the Poincareé dual to the fundamental class of nontwisted sector.*
2. *Restricted to each $H^d_{orb}(X;\mathcal{L}) \times H^{2n-d}_{orb}(X;\mathcal{L}) \to H^{2n}_{orb}(X;\mathcal{L}) = H^{2n}(X,\mathbf{C})$,*
$$\int_X \alpha \cup_{orb,\mathcal{L}} \beta = < \alpha, \beta >_{orb,\mathcal{L}} . \tag{2.20}$$
3. *The cup product $\cup_{orb,\mathcal{L}}$ is invariant under deformations of J.*
4. *When X is of integral degree shifting numbers, the total twisted orbifold cohomology group $H^*_{orb}(X;\mathcal{L})$ is integrally graded, and we have supercommutativity*
$$\alpha_1 \cup_{orb,\mathcal{L}} \alpha_2 = (-1)^{\deg\alpha_1 \cdot \deg\alpha_2} \alpha_2 \cup_{orb,\mathcal{L}} \alpha_1.$$
5. *Restricted to the nontwisted sectors, i.e., the ordinary cohomology $H^*(X;\mathbf{C})$, the cup product $\cup_{orb,\mathcal{L}}$ equals the ordinary cup product on X.*
6. *$\cup_{orb,\mathcal{L}}$ is associative.*

Similarly, we also have a holomorphic version.

Theorem 2.10: *Let X be an n-dimensional closed complex orbifold with complex structure J. The orbifold cup product*

$$\cup_{orb,\mathcal{L}} : H^{p,q}_{orb}(X;\mathcal{L}) \times H^{p',q'}_{orb}(X;\mathcal{L}) \to H^{p+p',q+q'}_{orb}(X;\mathcal{L})$$

has the following properties:

1. *The total orbifold Dolbeault cohomology group is a ring with unit $e^0_X \in H^{0,0}_{orb}(X;\mathcal{L})$ under $\cup_{orb,\mathcal{L}}$, where e^0_X is the class represented by the equal one constant function on X.*
2. *Restricted to each $H^{p,q}_{orb}(X;\mathcal{L}) \times H^{n-p,n-q}_{orb}(X;\mathcal{L}) \to H^{n,n}_{orb}(X;\mathcal{L})$, the integral $\int_X \alpha \cup_{orb,\mathcal{L}} \beta$ equals the Poincare pairing $< \alpha, \beta >_{orb,\mathcal{L}}$.*
3. *The cup product $\cup_{orb,\mathcal{L}}$ is invariant under the deformation of J.*
4. *When X is of integral degree shifting numbers, the total twisted orbifold Dolbeault cohomology group of X is integrally graded, and we have supercommutativity*
$$\alpha_1 \cup_{orb,\mathcal{L}} \alpha_2 = (-1)^{\deg \alpha_1 \cdot \deg \alpha_2} \alpha_2 \cup_{orb,\mathcal{L}} \alpha_1.$$
5. *Restricted to the nontwisted sectors, i.e., the ordinary Dolbeault cohomologies $H^{*,*}(X;\mathbf{C})$, the cup product $\cup_{orb,\mathcal{L}}$ equals the ordinary wedge product on X.*
6. *The cup product is associative.*
7. *When X is Kähler, the cup product $\cup_{orb,\mathcal{L}}$ coincides with the twisted orbifold cup product over the twisted orbifold cohomology groups $H^*_{orb}(X;\mathcal{L})$ under the relation*
$$H^r_{orb}(X;\mathcal{L}) \otimes \mathbf{C} = \oplus_{p+q=r} H^{p,q}_{orb}(X;\mathcal{L}).$$

Remark 2.11: *If X is open, we can define usual twisted orbifold cohomology $H^*_{orb}(X,\mathcal{L})$ and twisted orbifold cohomology with compact support $H^*_{orb,c}(X,\mathcal{L})$ in the same fashion. The Poincare paring should be understood as the paring between $H^d_{orb}(X,\mathcal{L})$ and $H^{2n-d}_{orb,c}(X,\mathcal{L})$.*

3 Orbifold fundamental group and discrete torsion

First, we recall the definition of orbifold fundamental group.

Definition 3.1: *A smooth map $f : Y \to X$ is an orbifold cover iff (1) each $p \in Y$ has a neighborhood U_p/G_p such that the restriction of f to U_p/G_p is isomorphic to a map $U_p/G_p \to U_p/\Gamma$ such that $G_p \subset \Gamma$ is a subgroup. (2) Each $q \in X$ has a neighborhood U_q/G_q for which each component of $f^{-1}(U_q/G_q)$ is isomorphic to U_q/Γ' such that $\Gamma' \subset G_q$ is a subgroup. An orbifold universal cover $f : Y \to X$ of X has the property: (i) Y is connected; (ii)if $f' : Y' \to X$ is an orbifold cover, then there exists an orbifold cover $h : Y \to Y'$ such that $f = f' \circ h$. If Y exists, we call Y the orbifold universal cover of X and the group of deck translations the orbifold fundamental group $\pi^{orb}_1(X)$ of X.*

By Thurston [T], an orbifold universal cover exists. It is clear from the definition that the orbifold universal cover is unique. Suppose that $f : Y \to X$ is an orbifold universal cover. Then

$$f : Y - f^{-1}(\Sigma X) \to X - \Sigma X \tag{3.1}$$

is an honest cover with $G = \pi_1^{orb}(X)$ as covering group, where Σ is the singular loci of X. Therefore, $X = Y/G$ and there is a surjective homomorphism

$$p_f : \pi_1(X - \Sigma X) \to G. \tag{3.2}$$

In general, (3.1) is not a universal covering. Hence, p_f is not an isomorphism.

Remark 3.2: *Suppose that $X = Z/G$ for an orbifold Z and Y is the orbifold universal cover of Z. By the definition, Y is an orbifold universal cover of X. It is clear that there is a short exact sequence*

$$1 \to \pi_1(Z) \to \pi^{orb}(X) \to G \to 1. \tag{3.3}$$

Example 3.3: Consider the Kummer surface T^4/τ where τ is the involution

$$\tau(e^{it_1}, e^{it_2}, e^{it_3}, e^{it_4}) = (e^{-it_1}, e^{-it_2}, e^{-it_3}, e^{-it_4}). \tag{3.4}$$

The universal cover is $\mathbf{R}^4$. The group G of deck translations is generated by translations λ_i by an integral point and the involution

$$\tau : (t_1, t_2, t_3, t_4) \to (-t_1, -t_2, -t_3, -t_4).$$

It is easy to check that

$$G = \{\lambda_i (i = 1, 2, 3, 4), \tau | \tau^2 = 1, \tau\lambda_i = \lambda_i^{-1}\tau, \} \tag{3.5}$$

where λ_i represents translation and τ represents involution.

Example 3.4: Let $T^6 = \mathbf{R}^6/\Gamma$ where Γ is the lattice of integral points. Consider $\mathbf{Z}_2^2$ acting on T^6 lifted to an action on $\mathbf{R}^6$ as

$$\sigma_1(t_1, t_2, t_3, t_4, t_5, t_6) = (-t_1, -t_2, -t_3, -t_4, t_5, t_6)$$

$$\sigma_2(t_1, t_2, t_3, t_4, t_5, t_6) = (-t_1, -t_2, t_3, t_4, -t_5, -t_6)$$

$$\sigma_3(t_1, t_2, t_3, t_4, t_5, t_6) = (t_1, t_2, -t_3, -t_4, -t_5, -t_6).$$

This example was considered by Vafa-Witten [VW]. The orbifold fundamental group

$$\pi_1^{orb}(T^6/\mathbf{Z}_2^2) = \{\tau_i (1 \leq i \leq 6), \sigma_j (1 \leq j \leq 3) |$$

$$\sigma_i^2 = 1, \sigma_1\tau_i = \tau_i^{-1}\sigma_1 (i \neq 5, 6), \sigma_2\tau_i = \tau_i^{-1}\sigma_2 (i \neq 3, 4), \sigma_3\tau_i = \tau_i^{-1}\sigma_3 (i \neq 1, 2)\}. \tag{3.6}$$

The following example was taken from [SC]

Example 3.5: Consider the orbifold Riemann surface Σ_g of genus g and n-orbifold points $\mathbf{z} = (x_1, \cdots, x_n)$ with orders $k_1, \cdots, k_n$. Then,

$$\pi_1^{orb}(\Sigma_g) = \{\lambda_i (i \leq 2g), \sigma_i (i \leq n) | \sigma_1 \cdots \sigma_n \prod_i [\lambda_{2i-1}, \lambda_{2i}] = 1, \sigma_i^{k_i} = 1\}, \tag{3.7}$$

where λ_i are the generators of $\pi_1(\Sigma_g)$ and σ_i are the generators of $\Sigma_g - \mathbf{z}$ represented by a loop around each orbifold point.

Note that $\pi_1^{orb}(\Sigma_g)$ is just $\pi_1(\Sigma_g - \mathbf{z})$ modulo by the relation $\sigma_i^{k_i} = 1$. This suggests that one can first take the cover of $\Sigma_g - \mathbf{z}$ induced by $\pi_1^{orb}(\Sigma)$. The relation $\sigma_i^{k_i} = 1$ implies that the preimage of the punctured disc around x_i is a punctured disc. Then we can fill in the center point to obtain the orbifold universal cover.

Definition 3.6: Suppose that $S \subset X$ is a connected component of singular loci. A local discrete torsion α_S at S is defined as a cohomology class $\alpha_S \in H^2(\pi_1^{orb}(U(S)), U(1)) = H^2(B\pi_1^{orb}(U(S)), U(1))$, where $U(S)$ is a small open neighborhood of S. A global discrete torsion $\alpha = \{\alpha_S\}$ is an assignment of a local discrete torsion to each connected component of singular loci.

If $X = Z/G$ for a finite group G, by Remark 3.2, there is a surjective homomorphism

$$\pi : \pi_1^{orb}(X) \to G.$$

π induces a homomorphism

$$\pi^* : H^2(G, U(1)) \to H^2(\pi_1^{orb}(X), U(1)). \tag{3.8}$$

Hence, an element of $H^2(G, U(1))$ induces a discrete torsion of X.

They are many ways to define $H^2(G, U(1))$. The definition $H^2(G, U(1)) = H^2(BG, U(1))$ is a very useful definition for computation since we can use algebro-topological machinery. However, we can also take the original definition in terms of cocycles. A 2-cocycle is a map $\alpha : G \times G \to U(1)$ satisfying

$$\alpha_{g,1} = \alpha_{1,g} = 1, \alpha_{g,hk}\alpha_{h,k} = \alpha_{g,h}\alpha_{gh,k}, \tag{3.9}$$

for any $g, h, k \in G$. We denote the set of two cocycles by $Z^2(G, U(1))$. For any map $\rho : G \to U(1)$ with $\rho_1 = 1$, its coboundary is defined by formula

$$(\delta\rho)_{g,h} = \rho_g \rho_h \rho_{gh}^{-1}. \tag{3.10}$$

Let $B^2(G, U(1))$ be the set of coboundaries. Then, $H^2(G, U(1)) = Z^2(G, U(1))/B^2(G, U(1))$. $H^2(G, U(1))$ naturally appears in many important places of mathematics. For example, it classifies the group extension of G by $U(1)$. If we have a unitary projective representation of G, it naturally induces a class of $H^2(G, U(1))$. In many instances, this class completely classifies the projective unitary representation. In fact, it is in this context that discrete torsion arises in orbifold string theory.

Definition 3.7: *For each 2-cocycle α, we define its phase*

$$\gamma(\alpha)_{g,h} = \alpha_{g,h}\alpha_{h,g}^{-1}. \tag{3.11}$$

It is clear that $\gamma(\alpha)_{g,g} = 1, \gamma(\alpha)_{g,h} = \gamma(\alpha)_{h,g}^{-1}$.

Lemma 3.8: *Suppose that $gh = hg, gk = kg$. Then*

(1) $\gamma(\delta\rho)_{g,h} = 1$.

(2) $\gamma(\alpha)_{g,hk} = \gamma(\alpha)_{g,h}\gamma(\alpha)_{g,k}$.

The (2) implies $L^\alpha_g = \gamma_{g,\cdot} : C(g) \to U(1)$ is a group homomorphism. We call L^α_g a α-twisted character.

Proof: (1) is obvious. For (2),

$$\begin{aligned}
\gamma(\alpha)_{g,hk} &= \alpha_{g,hk}\alpha^{-1}_{hk,g} \\
&= \alpha_{g,hk}\alpha^{-1}_{gh,k}\alpha_{hg,k}\alpha^{-1}_{h,gk}\alpha_{h,kg}\alpha^{-1}_{hk,g} \\
&= \alpha_{g,h}\alpha^{-1}_{h,k}\alpha_{g,k}\alpha^{-1}_{h,g}\alpha_{h,k}\alpha^{-1}_{k,g} \\
&= \gamma(\alpha)_{g,h}\gamma(\alpha)_{g,k}
\end{aligned}$$

Next, we calculate discrete torsions for some groups. We first consider the case of finite abelian group G. In this case $H^i(G, \mathbf{Q}) = 0$ for $i \neq 0$. The exact sequence

$$0 \to \mathbf{Z} \to \mathbf{C} \to \mathbf{C}^* \to 1$$

implies that $H^2(G, U(1)) = H^2(G, \mathbf{C}^*) = H^3(G, \mathbf{Z})$. By universal coefficient theorem, $H^3(G, \mathbf{Z}) = H_2(G, \mathbf{Z})$.

Example 3.9 $G = \mathbf{Z}/n \times \mathbf{Z}/m$**:** Notes that $H^2(G, U(1)) = H_2(G, \mathbf{Z}) = \mathbf{Z}/n \otimes \mathbf{Z}/m = Z_{gcd(n,m)}$. In this case, one can write down the phase of discrete torsion explicitly [VW]. Let $\xi(\zeta)$ be $n(m)$-root of unity. Any element of $\mathbf{Z}/n \times \mathbf{Z}/m$ can be written as (ξ^a, ζ^b). Let $p = gcd(n, m)$. The phase of a discrete torsion can be written as

$$\gamma_{(\xi^a,\zeta^b),(\xi^{a'},\zeta^{b'})} = \omega_p^{m(ab'-ba')}$$

with $\omega_p = e^{2\pi i/p}, m = 1, \cdots, p$. There are p-phases for p-discrete torsions. It is trivial to generalize this construction to an arbitrary finite abelian group.

4 Discrete torsion and local system

Suppose that $f : Y \to X$ is the orbifold universal cover and G is the orbifold fundamental group which acts on Y such that $X = Y/G$. Suppose $X_{(g)}$ is a sector (twisted or nontwisted) of X. For any $q \in X$, choose an orbifold chart U_q/G_q satisfying Definition 3.1. A component of $f^{-1}(U_q/G_q)$ is of the form U_q/Γ' for $\Gamma' \subset G_q$. It is clear that G_q/Γ' is a subgroup of the orbifold fundamental group. Therefore, we obtain a group homomorphism

$$\phi_q : G_q \to \pi_1^{orb}(X). \tag{4.1}$$

It is easy to check that a different choice of component of $f^{-1}(U_q/G_q)$ or a different choice of $q \in X_{(g)}$ induces a homomorphism differing by a conjugation. Therefore, there is a unique map from the conjugacy classes of G_q to the conjugacy classes of $\pi_1^{orb}(X)$.

Definition 4.1: *We call $X_{(g)}$ a dormant sector if $\phi_p(g) = 1$.*

If $X_{(g)}$ is a dormant sector, we define $L_{(g)} = 1$. It will not receive any correction from discrete torsion. Non-dormant sectors are of the form $Y_g/C(g)$, where $Y_g \neq \emptyset$ is the fixed point loci of $1 \neq g \in \pi_1^{orb}(X)$. Y_g is a smooth suborbifold of Y. It is clear that $Y_{h^{-1}gh}$ is diffeomorphic to Y_g by the action of h. By abusing the notation, we denote the twisted sector $Y_g/C(g)$ by $X_{(g)}$, where $C(g)$ is the centralizer of g.

Let α be a global discrete torsion. Suppose that S is the connected component of singular loci containing the image of $X_{(g)}$ in X. We choose a small open neighborhood $U(S)$ of S and suppose

that local discrete torsion is α_S. We replace X by $U(S)$ in above construction. We can use L_g^α to define a flat complex orbifold line-bundle

$$L_g = Y_g \times_{L_g^\alpha} \mathbf{C}$$

over $X_{(g)}$.

Lemma 4.2:

(1) $L_{tgt^{-1}}$ *is isomorphic to* L_g *by the map*

$$t \times Id : Y_g \times \mathbf{C} \to Y_{tgt^{-1}} \times \mathbf{C}. \tag{4.3}$$

Hence, we can denote L_g *by* $L_{(g)}$.

(2) $L_{(g)}^{-1} = L_{(g^{-1})}$.

(3) *When we restrict to* $X_{(g_1,\cdots,g_k)} = Y_{g_1} \cap \cdots \cap Y_{g_k}/C(g_1,\cdots,g_k)$, $L_{(g_1,\cdots,g_k)} = L_{(g_1)} \cdots L_{(g_k)}$, *where* $L_{(g_1,\cdots,g_k)} = Y_{g_1} \cap \cdots Y_{g_k} \times_{\gamma_{g_1\cdots g_k}} \mathbf{C}$.

Proof: Recall that there is an isomorphism

$$t_\# : C(g) \to C(tgt^{-1})$$

given by $t_\#(h) = tht^{-1}$. The map

$$t : Y_g \to X_{tgt^{-1}}$$

is $t_\#$-equivariant. By Lemma 3.8, $\gamma_{tgt^{-1}}(tht^{-1}) = \gamma_g(h)$ for $h \in C(g)$. Then,

$$(t \times Id)(hx, \gamma(h)(v)) = (thx, \gamma_g(h)(v)) = (tht^{-1}tx, \gamma_{tgt^{-1}}(tht^{-1})(v)). \tag{4.4}$$

Then we take the quotient by $C(g), C(tgt^{-1})$ respectively to get an isomorphism between $L_g, L_{tgt^{-1}}$. (2) and (3) follow from the fact that for any $h \in C(g_1,\cdots,g_k)$,

$$\gamma(\alpha)_{g_1\cdots g_k,h} = \gamma(\alpha)^{-1}_{h,g_1\cdots g_k} = \gamma(\alpha)^{-1}_{h,g_1} \cdots \gamma(\alpha)^{-1}_{h,g_k} = \gamma(\alpha)_{g_1,h} \cdots \gamma(\alpha)_{g_k,h}. \tag{4.5}$$

Theorem 4.3: $\mathcal{L}_\alpha = \{L_{(g)}\}_{(g)\in T_1}$ *is an inner local system of* X.

Proof: Property (1) is obvious. The property (2) follows from Lemma 4.2. Let's prove property (3). Consider the image $\mathbf{g}' = (g_1', g_2', g_3')$ of $\mathbf{g}$ in $\pi_1^{orb}(X)$ under the homomorphism (4.1). Then, we still have $g_1'g_2'g_3' = 1$. There are three possibilities, (i) $g_1' = g_2' = g_3' = 1$ and there is nothing to prove in this case; (ii) $g_3' = 1, g_2' = (g')_1^{-1}$ is nontrivial; (iii) g_1', g_2', g_3' are all nontrivial.

For the second case, let $g = g_1'$. We have the following factorization

$$e_1 \times e_2 \times e_3 : X_{(\mathbf{g})} \to X_{(g_1,g_2)} \times X_{(g_3)} \to X_{(g_1)} \times X_{(g_2)} \times X_{(g_3)}.$$

However, $X_{(g_1,g_2)} = Y_g \cap Y_{(g^{-1})}/C(g,g^{-1}) = Y_g/C(g)$. Moreover, over X_{g_1,g_2}

$$e_1^*L_{(g)}e_2^*L_{(g^{-1})} = L_{(g)}I^*L_{(g^{-1})} = 1. \tag{4.6}$$

In the third case, $X_{(\mathbf{g})} = Y_{g_1} \cap Y_{g_2} \cap Y_{g_3}/C(g_1,g_2,g_3)$. The proof follows from Lemma 4.2 (3).

Definition 4.4: *Suppose that* α *is a global discrete torsion. We define the twisted orbifold cohomology* $H^*_{orb,\alpha}(X,\mathbf{C}) = H^*_{orb}(X,\mathcal{L}_\alpha)$.

5 Examples

Only a few examples of global quotients have been computed by physicists [VW] [D]. It is still a very important problem to develop general machinery to compute discrete torsion and twisted orbifold cohomology. Here we compute five examples. First two have nontrivial discrete torsion. One is a global quotient and another one is a non-global quotient. The second example has the phenomenon that the most of twisted sectors are dormant sectors. The third one is Joyce example, where there is no nontrivial discrete torsion. However, there are nontrivial local systems. We will compute twisted orbifold cohomology given by nontrivial local systems to match Joyce's desingularizations. Orbifold cohomology is strongly intertwine with group theory. We demonstrate it in last two examples.

Example 5.1 $T^4/\mathbf{Z}_2 \times \mathbf{Z}_2$**:** Here, $T^4 = \mathbf{C}^2/\wedge$, where $\wedge$ is the lattice of integral points. Suppose that g, h are generators of the first and the second factor of $\mathbf{Z}_2 \times \mathbf{Z}_2$. The action of $\mathbf{Z}_2 \times \mathbf{Z}_2$ on T^4 is defined as

$$g(z_1, z_2) = (-z_1, z_2), h(z_1, z_2) = (z_1, -z_2). \tag{5.1}$$

The fixed point locus of g is 4 copies of T^2. When we divide it by the remaining action generated by h, we obtain twisted sectors consisting of 4 copies of S^2. The degree shifting number for these twisted sectors is $\frac{1}{2}$. For the same reason, the fixed point locus of h give twisted sectors consisting of 4 copies of S^2 with degree shifting number $\frac{1}{2}$. The fixed point locus of gh is 16 points, which are fixed by the whole group. The degree shifting number of the 16 points is 1. An easy calculation shows that nontwisted sector contributes one generator to degree 0, 4 orbifold cohomology and two generators to degree 2 orbifold cohomology and no other. Using this information, we can compute the ordinary orbifold cohomology group

$$b^0_{orb} = b^4_{orb} = 1, b^1_{orb} = b^3_{orb} = 8, b^2_{orb} = 18. \tag{5.2}$$

By example 2.10, $H^2(\mathbf{Z}_2 \times \mathbf{Z}_2, U(1)) = \mathbf{Z}_2$. By Remark 2.2, the nontrivial generator of $H^2(\mathbf{Z}_2 \times \mathbf{Z}_2, U(1))$ induces a discrete torsion α. Next, we compute the twisted orbifold cohomology $H^*_{orb,\alpha}(T^4/\mathbf{Z}_2 \times \mathbf{Z}_2, \mathbf{C})$. Note that $\gamma(\alpha)_{gh,g} = \gamma(\alpha)_{gh,h} = -1$. Hence, the flat orbifold-bundles over the twisted sectors given by 16 fixed points of gh are nontrivial. Therefore, they contribute nothing to twisted orbifold cohomology. For two dimensional twisted sectors, let's consider a component of fixed point locus of g. By the previous description, it is T^2. h acts on T^2. Then the twisted sector $S^2 = T^2/\{h\}$. We observe that the flat orbifold line bundle over S^2 is constructed as $L = T^2 \times_{L^\alpha_g} \mathbf{C}$. Hence $H^*(S^2, L)$ is isomorphic to the space of invariant cohomology of T^2 under the action of h twisted by $\gamma(\alpha)_g$ as $h(\beta) = \gamma(\alpha)_{g,h} h^*\beta$. By example 2.10, $\gamma(\alpha)_{g,h} = -1$. The invariant cohomology is $H^1(T^2, \mathbf{C})$. Using the degree shifting number to shift up its degree, we obtain the twisted orbifold cohomology

$$b^0_{orb,\alpha} = b^4_{orb,\alpha} = 1, b^1_{orb,\alpha} = b^3_{orb,\alpha} = 0, b^2_{orb,\alpha} = 18. \tag{5.3}$$

Example 5.2 $WP(2, 2d_1) \times WP(2, 2d_2)$ **(**$d_1, d_2 > 1, (d_1, d_2) = 1$**):** Here, $WP(2, 2d)$ is the weighted projective space of weighted $(2, 2d)$. $WP(2, 2d_1) \times WP(2, 2d_2)$ is not a global quotient unless $d_1 = d_2 = 1$. In fact, its orbifold universal cover is $WP(1, d_1) \times WP(1, d_2)$ and $WP(2, 2d_1) \times WP(2, 2d_2) = WP(1, d_1) \times WP(1, d_2)/\mathbf{Z}_2 \times \mathbf{Z}_2$. Hence, the orbifold fundamental group is $\mathbf{Z}_2 \times \mathbf{Z}_2$. Therefore, there is a nontrivial discrete torsion $\alpha \in H^2(\mathbf{Z}_2 \times \mathbf{Z}_2, U(1))$.

Next, we describe the twisted sectors. Suppose that $p = [0, 1], q = [1, 0] \in WP(1, d_1)$. We also use p, q to denote its image in $WP(2, 2d_1)$. We use p', q' to denote the corresponding points in

$WP(1,d_2), WP(2,2d_2)$. $\{p\}\times WP(2,2d_2), \{p'\}\times WP(2,2d_1)$ give rise to two twisted sectors with degree shifting number $\frac{1}{2}$. $\{q\}\times WP(2,2d_2), \{q'\}\times WP(2,2d_1)$ give rise to $2d_1-1, 2d_2-1$ many twisted sectors with degree shifting numbers $\frac{i}{2d_1}, \frac{j}{2d_2}$ for $1\leq i\leq 2d_1-1, 1\leq j\leq 2d_2-1$. $\{p\}\times\{p'\}$ give rise to a twisted sector with degree shifting number 1. $\{p\}\times\{q'\}$ give rise to $2d_2-1$-many twisted sectors with degree shifting numbers $\frac{1}{2}+\frac{i}{2d_2}$ for $1\leq i\leq 2d_2-1$. $\{q\}\times\{p'\}$ give rise to $2d_1-1$-many twisted sectors with degree shifting numbers $\frac{1}{2}+\frac{i}{2d_1}$ for $1\leq i\leq 2d_1-1$. $\{q\}\times\{q'\}$ give rise to $4d_1d_2-1$-many twisted sectors with degree shifting numbers $\frac{i}{2d_1}+\frac{j}{2d_2}$ for all i,j except $(i,j)=(0,0)$. Using this information, we can write down ordinary orbifold cohomology

$$b^0_{orb}=b^4_{orb}=1, b^1_{orb}=b^3_{orb}=6, b^2_{orb}=6$$

$$b^{\frac{i}{d_1}}_{orb}=b^{\frac{i}{d_2}}_{orb}=1, b^{1+\frac{i}{d_1}}_{orb}=b^{1+\frac{i}{d_2}}_{orb}=3, b^{2+\frac{i}{d_1}}_{orb}=b^{2+\frac{i}{d_2}}_{orb}=2,\ 1\leq i\leq d_1-1, 1\leq j\leq d_2-1$$

$$b^{\frac{i}{d_1}+\frac{j}{d_2}}_{orb}=1, 0\leq i\leq 2d_1-1, 0\leq j\leq 2d_2, (i,j)\neq(0,0),(d_1,d_2). \tag{5.4}$$

Next, we compute $H^*_{orb,\alpha}$. In this example, the most of twisted sectors are dormant sectors. To find nondormant sectors, recall that $WP(2,2d_1)\times WP(2,2d_2)=WP(1,d_1)\times WP(1,d_2)/Z_2\times Z_2$. Let g be the generator of the first factor and h be the generator of the second factor. The fixed points of g is $\{p,q\}\times WP(1,d_2)$. We have two nondormant sectors obtained by modulo the remaining action generated by h. However, $\gamma(\alpha)_{g,h}=-1$. There is no invariant cohomology of $WP(1,d_2)$ under the action of h twisted by L^{α}_g. Hence, these two nondormant twisted sectors give no contribution to twisted orbifold cohomology. Their degree shifting numbers are 1. For the same reason, $WP(1,d_1)\times\{p',q'\}/g$ gives no contribution to twisted orbifold cohomology. The fixed point locus of gh consists of 4 points which give 4 nondormant sectors. Again, their degree shifting numbers are 1. As we saw in last example, their flat orbifold bundles are nontrivial. Hence, they give no contribution to twisted orbifold cohomology. Therefore, the twisted orbifold cohomology is

$$b^0_{orb,\alpha}=b^4_{orb,\alpha}=1, b^1_{orb,\alpha}=b^3_{orb,\alpha}=2, b^2_{orb,\alpha}=2$$

$$b^{\frac{i}{d_1}}_{orb,\alpha}=b^{\frac{i}{d_2}}_{orb,\alpha}=1, b^{1+\frac{i}{d_1}}_{orb,\alpha}=b^{1+\frac{i}{d_2}}_{orb,\alpha}=3, b^{2+\frac{i}{d_1}}_{orb}=b^{2+\frac{i}{d_2}}_{orb,\alpha}=2,\ 1\leq i\leq d_1-1, 1\leq j\leq d_2-1$$

$$b^{\frac{i}{d_1}+\frac{j}{d_2}}_{orb,\alpha}=1,\ 0\leq i\leq 2d_1-1, 0\leq j\leq 2d_2, (i,j)\neq(0,0),(d_1,d_2). \tag{5.5}$$

Example 5.3 $T^6/\mathbf{Z}_4$**:** Here, $T^6=\mathbf{C}^3/\wedge$, where $\wedge$ is the lattice of integral points. The generator of $\mathbf{Z}_4$ acts on T^6 as

$$\kappa:(z_1,z_2,z_3)\to(-z_1, iz_2, iz_3). \tag{5.6}$$

This example has been studied by D. Joyce [JO], where he constructed five different desingularizations. However, there is no discrete torsion in the case which induces nontrivial orbifold cohomology.

First all, the nontwisted sector contributes one generator to $H^{0,0}_{orb}, H^{3,3}_{orb}$, 5 generators to $H^{1,1}_{orb}, H^{2,2}_{orb}$ and 2 generator to $H^{2,1}_{orb}, H^{1,2}_{orb}$ The fixed point loci of κ,κ^3 are 16-points

$$\{(z_1,z_2,z_3)+\wedge: z_1\in\{0,\frac{1}{2},\frac{i}{2},\frac{1}{2}+\frac{i}{2}\}, z_2,z_3\in\{0,\frac{1}{2}+\frac{i}{2}\}.$$

These points are fixed by Z_4. Therefore, they generate 32-twisted sectors in which 16 corresponds to the conjugacy class (κ) and 16 corresponds to the conjugacy class (κ^3). The sector with conjugacy class (κ) has degree shifting number 1. The sector with conjugacy class (κ^3) has degree shifting number 2.

The fixed point loci of κ^2 is 16 copies of T^2, given by

$$\{(z_1, z_2, z_3) + \wedge : z_1 \in \mathbf{C}, z_2, z_3 \in \{0, \frac{1}{2}, \frac{i}{2}, \frac{1}{2} + \frac{i}{2}\}\}$$

Twelve of the 16-copies of T^2 fixed by κ^2 are identified in pairs by the action of κ, and these contribute 6 copies of T^2 to the singular set of T^6/Z_4. On the remaining 4 copies κ acts as -1, so these contribute 4 copies of $S^2 = T^2/\{\pm 1\}$ to singular set. The degree shifting number of these 2-dimensional twisted sectors is 1.

Next, we construct inner local systems. We start with two dimensional twisted sectors. Since $\kappa^{-2} = \kappa^2$, the condition (2) of Definition 2.1 tells us that the flat orbifold line bundle L over two dimensional sectors has the property $L^2 = 1$. Now, we assign trivial line bundle to all T^2-sectors and $k(k = 0, 1, 2, 3, 4)$-many $S^2 = T^2/\{\pm 1\}$-sectors. For the remaining $S^2 = T^2/\{\pm 1\}$-sectors, we assign a flat orbifold line bundle $T^2 \times \mathbf{C}/\{\pm 1\}$. For the zero dimensional sectors, they are all points of two dimensional sectors. If we assign a trivial bundle on a two dimensional sector, we just assign trivial bundle to its point sectors. For these two dimensional sectors with nontrivial flat line bundle, we need to be careful to choose the flat orbifold line bundle on its point sectors to ensure the condition (3) of Definition 2.1. Suppose that Σ is one of 2-dimensional sectors supporting nontrivial flat orbifold line bundle. It contains 4 singular points which generate the point sectors. Let x be one of 4-points. x generates two sectors given by the conjugacy classes $(\kappa), (\kappa^3)$. For condition (3), we have to consider the conjugacy class of triple (g_1, g_2, g_3) with $g_1 g_2 g_3 = 1$. The only nontrivial choices are $(\mathbf{g}) = (\kappa, \kappa, \kappa^2), (\kappa^2, \kappa^3, \kappa^3)$. The corresponding components of $X_{(\mathbf{g})}$ are exactly the these singular points. Clearly, x is a fixed by the whole group Z_4. The orbifold line bundle is given by the action of $\mathbf{Z}_4$ on $\mathbf{C}$. Consider the component of $X_{(\mathbf{g})}$ generated by x. The pull-back of flat orbifold line bundle from 2-dimensional sector ((κ^2)-sector) is given by the action $\kappa v = -1$. A moment of thought tells us that for sectors $(\kappa), (\kappa^3)$, we should assign a flat orbifold line bundle given by the action of $\mathbf{Z}_4$ on $\mathbf{C}$ as $\kappa v = iv$. It is easy to check that for above choices the condition (3) is satisfied for $X_{(\mathbf{g})}$. Therefore, the twisted sectors given by $(x, (\kappa)), (x, (\kappa^3))$ give no contribution to twisted orbifold cohomology. Suppose that the resulting local system is $\mathcal{L}_k$. For the sectors with trivial line bundle, they contribute $6 + k$ generators to $H^{1,1}_{orb}, H^{2,2}_{orb}$ and 6 generators to $H^{2,1}_{orb}, H^{1,2}_{orb}$. Its point sectors contribute $4k$ generators to $H^{1,1}_{orb}, H^{2,2}_{orb}$. The remaining sectors contribute $4 - k$ generators to $H^{2,1}_{orb}, H^{1,2}_{orb}$. Its point sectors give no contribution. Moreover, the nontwisted sector contributes

$$h^{0,0} = h^{3,3} = 2, h^{1,1} = 5.$$

In summary, we obtain

$$\dim H^{0,0}_{orb}(T^6/Z_4, \mathcal{L}_k) = \dim H^{3,3}_{orb}(T^6/Z_4, \mathcal{L}_k) = 1, \dim H^{1,1}_{orb}(T^6/Z_4, \mathcal{L}_k) = \dim H^{2,2}_{orb}(T^6/Z_4, \mathcal{L}_k) = 11+5k,$$

$$\dim H^{1,2}_{orb}(T^6/Z_4, \mathcal{L}_k) = \dim H^{2,1}_{orb}(T^6/Z_4, \mathcal{L}_k) = 12 - k \tag{5.7}$$

Our calculation matches the betti numbers of Joyce's desingularizations.

The orbifold cohomology ring of following examples have been computed in [CR]. Here, we compute their twisted version.

Example 5.4: Let's consider the case that X is a point with a trivial group action of G. Suppose that $\alpha \in H^2(G, U(1))$ is a discrete torsion. We want to compute $H^*_{orb,\alpha}(X, \mathbf{C})$. The twisted sector $X_{(g)}$ is a point with a group $C(g)$. It is obvious that $H^0(X_{(g)}, L^{\alpha}_g) = 0$ unless $L^{\alpha}_g = 1$. Recall that a conjugacy class (g) is α-regular iff $L^{\alpha}_g = 1$. Hence, only α-regular class will contribute. Therefore, torbifold cohomology is generated by α-regular conjugacy classes of elements of G. All the degree shifting numbers are zero. By the same argument as nontwisted case, as a ring, $H^*_{orb,\alpha}(X, \mathbf{C})$ is the center of twisted group algebra $\mathbf{C}_\alpha[G]$.

Example 5.5: Suppose that $G \subset SL(n, \mathbf{C})$ is a finite subgroup. Then, $\mathbf{C}^n/G$ is an orbifold. Suppose that $\alpha \in H^2(G, U(1))$ is a discrete torsion. For any $g \in G$, the fixed point set X_g is a vector subspace and $X_{(g)} = X_g/C(g)$. By the definition, $L_{(g)} = X_g \times \gamma(\alpha)_g \mathbf{C}$. Therefore, $H^*(X_{(g)}, L_{(g)})$ is the subspace of $H^*(X_g, \mathbf{C})$ invariant under twisted action of $C(g)$

$$h \circ \beta = \gamma(\alpha)_g(h) h^* \beta \tag{5.8}$$

for any $h \in C(g), \beta \in H^*(X_g, \mathbf{C})$. However, $H^i(X_g, \mathbf{C}) = 0$ for $i \geq 1$. Moreover, if $\gamma(\alpha)_g$ is nontrivial, $H^0(X_g, L_{(g)}) = 0$. Therefore, $H^{p,q}_{orb} = 0$ for $p \neq q$ and $H^{p,p}_{orb}$ is a vector space generated by conjugacy class of α-regular elements g with $\iota_{(g)} = p$. Therefore, we have a natural decomposition

$$H^*_{orb,\alpha}(X, \mathbf{C}) = Z[\mathbf{C}_\alpha[G]) = \sum_p H_p, \tag{5.9}$$

where H_p is generated by conjugacy classes of α-regular elements g with $\iota_{(g)} = p$. The ring structure is also easy to describe. Let $x_{(g)}$ be generator corresponding to zero cohomology class of twisted sector $X_{(g)}$ such that g is α-regular. The cup product is completely same as the nontwisted case except we replace conjugacy class by α-conjugacy class. Let me sketch the calculation. As we showed in previous example, the multiplication of conjugacy classes can be described in terms of center of twisted group algebra $Z(\mathbf{C}_\alpha[G])$. But we have further restrictions in this case. It is clear

$$X_{(h_1, h_2, (h_1h_2)^{-1})} = X_{h_1} \cap X_{h_2}/C(h_1, h_2).$$

To have nonzero invariant, we require that

$$\iota_{(h_1h_2)} = \iota_{(h_1)} + \iota_{(h_2)}. \tag{5.10}$$

Then, we need to compute

$$\int_{X_{h_1} \cap X_{h_2}/C(h_1, h_2)} e_3^*(vol_c(X_{h_1h_2}/C(h_1h_2))) \wedge e(E), \tag{5.11}$$

where $vol_c(X_{h_1h_2}/C(h_1h_2))$ is the compact supported top form with volume one. However,

$$X_{h_1} \cap X_{h_2}/ \subset X_{h_1h_2}$$

is a submanifold. (5.11) is zero unless

$$X_{h_1} \cap X_{h_2} = X_{h_1h_2}. \tag{5.12}$$

In this case, we call (h_1, h_2) transverse. In this case, it is clear that obstruction bundle is trivial. Suppose that d_{h_1,h_2} is the order of finite cover $X_{h_1h_2}/C(h_1, h_2) \rightarrow X_{h_1h_2}/C(h_1h_2)$. Then, the integral is d_{h_1,h_2}. Let

$$I_{g_1,g_2} = \{(h_1, h_2); h_i \in (g_i), \iota_{(h_1)} + \iota_{(h_2)} = \iota_{(h_1h_2)}, (h_1, h_2) - transverse, (h_1h_2) - \alpha - regular\}. \tag{5.13}$$

Then,

$$x_{(g_1)} \cup x_{(g_2)} = \sum_{(h_1,h_2) \in I_{g_1,g_2}} d_{h_1,h_2} x_{(h_1h_2)}. \tag{5.14}$$

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